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Unicity and Strong Unicity in Approximation Theory

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1. INTRODUCTION

It is the object of this paper to discuss the following question: Is it possible to characterize unicity and strong unicity of elements of best approximation by modified Kolmogorov-criteria? Furthermore, we examine the relationship between these two properties.

Let G be a nonempty set in a normed linear space E, and let f be an element of E. Consider $P(f) := P_G(f) := \{g_0 \in G: \|f - g_0\| \leqslant \|f - g\|, g \in G\}$, i.e., the set of *elements of best approximation* of f in G. The set-valued map $P: E \to 2^G$ defined in this way is called the *metric projection*. The set G is called *proximinal* (respectively, *semi-Chebyshev*) if $P(f)$ is nonempty (respectively, contains at most one element) for each $f \in E$. If G is both proximinal and semi-Chebyshev, then it is called *Chebyshev*. For each $f \in E$ let $S_f := \{L \in E': \|L\| = 1, L(f) = \|f\|\}$ and let E_f be the set of extreme points of S_f in the $\sigma(E', E)$-topology. We say the pair (g_0, f), with $g_0 \in G$ and $f \in E \backslash \bar{G}$, satisfies the *Kolmogorov-criterion* if, for each $g \in G$,

$$\min \{\operatorname{Re} L(g - g_0) : L \in E_{f-g_0}\} \leqslant 0;$$

the strict Kolmogorov-criterion if, for each $g \in G$, $g \neq g_0$,

$$\min \{\operatorname{Re} L(g - g_0) : L \in E_{f-g_0}\} < 0;$$

and the *strong Kolmogorov-criterion* if there exists a constant $K > 0$ such that, for each $g \in G$,

$$\min \{\operatorname{Re} L(g - g_0) : L \in E_{f-g_0}\} \leqslant -K \|g - g_0\|.$$

Brosowski [3] proved that a set G in a normed linear space E is a sun if and only if for each $f \in E \backslash \bar{G}$ the element $g_0 \in G$ is in $P(f)$ if and only if the pair (g_0, f) satisfies the Kolmogorov-criterion. This result has led us to ask whether

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it is possible to characterize certain semi-Chebyshev sets G in an arbitrary normed linear space E by the strict Kolmogorov-criterion. In Section 3 we will see that this is in general not possible, not even if G is a finite-dimensional subspace of E. But we show in Section 2 that it can be done for finite-dimensional convex sets G in $E = l_A$, which includes the cases $E = l_\phi = C(X)$ and $E = l_{\{\infty\}} = C_0(T)$, and for suns G in $E = L_1(T, m)$ using characterizations of best approximations given by Brosowski [3], Deutsch [5], Deutsch–Maserick [6] and Havinson [9].

Furthermore in general it is not possible to characterize those elements f in E for which $P(f)$ is a singleton by the strict Kolmogorov-criterion, not even for the finite-dimensional subspace of the splinefunctions in $C[a, b]$. But we are able to verify in this case that under certain alternation properties on $f - g_0$ the pair (g_0, f) satisfies the strict Kolmogorov-criterion.

In Section 3 we show by using results of Bartelt, McLaughlin [2] and Wulbert [17] that strongly unique elements of best approximation (see Definition 3.1) can be characterized by the strong Kolmogorov-criterion in the case when G is a linear subspace in an arbitrary normed linear space E.

Finally we apply our theorems of Section 2 to obtain statements concerning strong unicity and pointwise Lipschitzian metric projections (see Definition 3.9), which include results of Ault, Deutsch, Morris, Olson [1], Freud [8], Newman, Shapiro [12], Schumaker [14] and Wulbert [17].

Notation. For a normed linear space E we denote by E' the dual space of E and by $S_E := \{f \in E: \|f\| \leqslant 1\}$ its unit ball. For a set A in E and a function f defined on E we denote the restriction of f to A by $f|_A$ and the extreme points of A by $Ep(A)$. We say that a set G in E is a convex cone, if G is closed, convex and $ag \in G$ for each $g \in G$ and $a \geqslant 0$. For a function f defined on a set T we denote $Z(f) := \{t \in T: f(t) = 0\}$.

2. Unicity of Best Approximations

The condition that the pair (g_0, f) satisfies the strict Kolmogorov-criterion is sufficient that g_0 is the only best approximation of f:

2.1. Lemma. *Let E be a normed linear space, G a nonempty subset of E, $f \in E\backslash\bar{G}$ and $g_0 \in G$. If (g_0, f) satisfies the strict Kolmogorov-criterion then $\{g_0\} = P(f)$.*

Proof. According to the assumption for each $g \in G$, $g \neq g_0$, there exists a functional $L_g \in E_{f-g_0}$ with $\operatorname{Re} L_g(g - g_0) < 0$. Then for each $g \in G$, $g \neq g_0$, $\|f-g\| \geqslant |L_g(f-g)| > \operatorname{Re}\ L_g(f-g) + \operatorname{Re}\ L(g-g_0) = \operatorname{Re}\ L_g(f-g_0) = \|f-g_0\|$. Clearly this implies $P(f) = \{g_0\}$.

In view of Lemma 2.1 it is natural to ask whether semi-Chebyshev sets G in a normed linear space can be characterized by the strict Kolmogorov-criterion. As we will see in Section 3 this is not possible even for finite-dimensional subspaces G in an arbitrary normed linear space E. But we are able to prove theorems of this type for certain normed linear spaces.

First we consider the case $E = I_A$: For a compact space X we denote by $C(X)$ the space of all real-valued, continuous functions on X endowed with the usual vector operations and with the norm $\|f\| = \max\{|f(x)|: x \in E\}$ for each f in $C(X)$.

For a locally compact space T we denote by $C_0(T)$ the space of all continuous functions on T, vanishing at infinity, endowed with the usual vector operations and with the norm

$$\|f\| = \sup\{|f(t)|: t \in T\} \text{ for each } f \text{ in } C_0(T).$$

If A is a closed set in X we denote by I_A the linear subspace

$$I_A = \{f \in C(X): f(x) = 0\ (x \in A)\} \text{ of } C(X).$$

In particular $I_\phi = C(X)$. Compactifying T by adding a point ∞ we obtain a compact Hausdorff space $X_0 = T \cup \{\infty\}$ and $C_0(T)$ may be considered as $I_{\{\infty\}} \subset C(X_0)$.

We need the following characterization for the elements of best approximation in some finite-dimensional convex set of a normed linear spaces given by Deutsch, Maserick [6] and, independently, by Havinson [9]:

2.2. THEOREM. *Let G be a convex set in a real normed linear space E, let $f \in E \backslash \bar{G}$, $g_0 \in G$ and suppose the span G is n-dimensional. Then $g_0 \in P(f)$ if and only if there exist m linear independent functionals $L_1, \dots, L_m \in E_{f-g_0}$ and m numbers $a_1, \dots, a_m > 0$ with $\sum_{i=1}^m a_i = 1$, where $1 \leqslant m \leqslant n+1$, such that*

$$\sum_{i=1}^{m} a_i L_i(g - g_0) \leqslant 0 \text{ for each } g \in G.$$

It is well known that for the case $E = I_A$ each $L \in E_{f-g_0}$ is of the form

$$L(h) = \frac{(f - g_0)(x)}{\|f - g_0\|} h(x) \text{ for each } h \in I_A$$

where $x \in M_{f-g_0} \backslash A$ and $M_{f-g_0} = \{x \in X: |(f - g_0)(x)| = \|f - g_0\|\}$. The converse is true for $I_\phi = C(X)$ (see Dunford-Schwartz [7]). Using this fact and Theorem 2.2 we have the following corollary:

2.3. COROLLARY. *Let G be a convex set of I_A, $f \in E \backslash \bar{G}$, $g_0 \in G$ and suppose the span G is n-dimensional. Then $g_0 \in P(f)$ if and only if there exist m distinct points $x_1, \ldots, x_m \in M_{f-g_0} \backslash A$ and m numbers $a_1, \ldots, a_m > 0$ with $\sum_{i=1}^{m} a_i = 1$ where $1 \leqslant m \leqslant n+1$, such that*

$$\sum_{i=1}^{m} a_i(f - g_0)(g - g_0)(x_i) \leqslant 0 \text{ for each } g \in G.$$

Using Corollary 2.3 we give a characterization of finite-dimensional convex semi-Chebyshev sets in I_A:

2.4. THEOREM. *Let G be a finite-dimensional convex set in $E = I_A$. Then the following statements are equivalent:*

(1) *G is semi-Chebyshev*

(2) *For each $f \in E \backslash \bar{G}$, $g_0 \in P(f)$, $g \in G$, $g \neq g_0$,*

$$\min \{(f - g_0)(g - g_0)(x) : x \in M_{f-g_0}\} < 0$$

If $E = I_\phi = C(X)$ the conditions (1) *and* (2) *are equivalent to the following statement:*

(3) *For each $f \in E \backslash \bar{G}$, $g_0 \in P(f)$ the pair (g_0, f) satisfies the strict Kolmogorov-criterion.*

Proof. Assume we have (2), then for $f \in E \backslash \bar{G}$, $g_0 \in P(f)$ and $g \in G$, $g \neq g_0$, there exists a point $x \in M_{f-g_0} \backslash A$ with $(f - g_0)(g - g_0)(x) < 0$. Therefore $\|f - g_0\| = |(f - g_0)(x)| < |(f - g_0)(x) - (g - g_0)(x)| = |(f - g)(x)| \leqslant \|f - g\|$. Thus $\{g_0\} = P(f)$. Proving (1).

We show that (2) follows from (1): Assume that there exist $f \in E \backslash \bar{G}$, $g_0 \in P(f)$ and $g_1 \in G$, $g_1 \neq g_0$, such that for each $x \in M_{f-g_0}$

$$(f - g_0)(g_1 - g_0)(x) \geqslant 0. \tag{a}$$

We show that there exists a function f_0 in E with $g_0, g_1 \in P(f_0)$. Since span G is n-dimensional and $g_0 \in P(f)$ by Corollary 3 there exist m distinct points $x_1, \ldots, x_m \in M_{f-g_0} \backslash A$ and m numbers $a_1, \ldots, a_m > 0$ with $\sum_{i=1}^{m} a_i = 1$, where $1 \leqslant m \leqslant n+1$, such that

$$\sum_{i=1}^{m} a_i(f - g_0)(g - g_0)(x_i) \leqslant 0 \text{ for each } g \in G. \tag{b}$$

Since $a_1, \ldots, a_m > 0$ it follows from (a) and (b)

$$(f - g_0)(g_1 - g_0)(x_i) = 0,\ 1 \leqslant i \leqslant m$$

and since $x_1,\dots,x_m \in M_{f-g_0}\backslash A$

$$(g_1 - g_0)(x_i) = 0, \qquad 1 \leqslant i \leqslant m. \tag{c}$$

We define for each $x \in X$

$$f_0(x) := \frac{(f-g_0)(x)}{\|f-g_0\|}(\|g_1-g_0\| - |(g_1-g_0)(x)|) + g_0(x).$$

Obviously f_0 is in E. Then

$$|(f-g_0)(x)| = \frac{|(f-g_0)(x)|}{\|f-g_0\|}(\|g_1-g_0\| - |(g_1-g_0)(x)|) \leqslant \|g_1-g_0\|.$$

and because of (c) and $x_1,\dots,x_m \in M_{f-g_0}\backslash A$

$$|(f_0-g_0)(x_i)| = \|g_1-g_0\|, \qquad 1 \leqslant i \leqslant m.$$

Therefore $\|f_0 - g_0\| = \|g_1 - g_0\|$.
Because of (b) and $a_1,\dots,a_m > 0$ for each $g \in G$ there exists a point $x_i \in M_{f-g_0}\backslash A$, $1 \leqslant i \leqslant m$, with

$$(f-g_0)(g-g_0)(x_i) \leqslant 0. \tag{d}$$

Therefore by (c) and (d)

$$\begin{aligned}\|f_0-g\| &\geqslant \left\| \frac{(f-g_0)(x_i)}{\|f-g_0\|}(\|g_1-g_0\| - |(g_1-g_0)(x_i)|) + (g_0-g)(x_i)\right\| \\ &= \left\| \frac{(f-g_0)(x_i)}{\|f-g_0\|}\|g_1-g_0\| + (g_0-g)(x_i)\right\| \\ &= \frac{|(f-g_0)(x_i)|}{\|f-g_0\|}\|g_1-g_0\| + |(g_0-g)(x_i)| \geqslant \|g_1-g_0\| \\ &= \|f_0-g_0\|.\end{aligned}$$

Therefore $g_0 \in P(f_0)$.
Moreover for each $x \in X$

$$\begin{aligned}&|(f_0-g_1)(x)| \\ &\quad = \left| \frac{(f-g_0)(x_i)}{\|f-g_0\|}(\|g_1-g_0\| - |(g_1-g_0)(x)|) + (g_0-g_1)(x)\right| \\ &\quad \leqslant \frac{|(f-g_0)(x)|}{\|f-g_0\|}(\|g_1-g_0\| - |(g_1-g_0)(x)|) + |(g_0-g_1)(x)| \\ &\quad \leqslant \|g_1-g_0\| - |(g_1-g_0)(x)| + |(g_0-g_1)(x)| = \|g_1-g_0\| \\ &\quad = \|f_0-g_0\|.\end{aligned}$$

Therefore $\|f_0 - g_1\| = \|f_0 - g_0\|$ and since $g_0 \in P(f)$ the function g_1 is in $P(f)$. The fact that $g_0, g_1 \in P(f)$ and $g_1 \neq g_0$ is a contradiction to G being semi-Chebyshev. Thus (1) implies (2).

The equivalence of (2) and (3) in the case $E = I_\phi = C(X)$ follows from the representation of the extreme points of the unitsphere in $C(X)'$.

Using Theorem 2.4 we can prove the following necessary condition for finite-dimensional convex sets G in I_A to be Chebyshev, for X metric.

2.5. COROLLARY. *Let G be an n-dimensional convex set in $E = I_A$, such that 0 is in G. If G is Chebyshev then each $g \in G$, $g \neq 0$, has at most $n - 1$ distinct zeros in $X \setminus A$.*

Proof. Assume that there exists a function $g_0 \in G$, $g_0 \neq 0$, with n distinct zeros $x_1, \ldots, x_n \in X \setminus A$. Then by a standard argument there exist n numbers $a_1, \ldots, a_n$ with $\sum_{i=1}^n |a_i| > 0$ such that for each $g \in G$ $\sum_{i=1}^n a_i g(x_i) = 0$. By Tietze's Lemma there exist a function $f \in E$ with $f(x_i) = \operatorname{sgn} a_i$ and $|f(x)| < 1$ elsewhere. Then for each $g \in G$ $\sum_{i=1}^n |a_i| fg(x_i) = \sum_{i=1}^n |a_i| \operatorname{sgn} a_i g(x_i) = \sum_{i=1}^n a_i g(x_i) = 0$. Replacing, if necessary, each $|a_i|$ by $|a_i| / \sum_{i=1}^n |a_i|$ we may assume that $\sum_{i=1}^n |a_i| = 1$.

Therefore by Corollary 2.3 the function 0 is in $P(f)$. Moreover $M_f = \{x_1, \ldots, x_n\}$ and for each $x \in M_f$ $fg_0(x) = 0$. By Theorem 2.4 it follows that G is not Chebyshev.

Corollary 2.5 has been proved by Phelps [13] for n-dimensional subspaces of I_A. The converse of Corollary 2.5 does not hold, as can be seen by easily constructed examples in $C(\{1, 2\})$.

Now we consider the case $E = L_1(T, m)$: For a positive measure space (T, m) we denote by $L_1(T, m)$ (respectively by $L_\infty(T, m)$) the space of all equivalence classes of m-integrable (respectively m-measurable and m-essentially bounded) real-valued functions on T, endowed with the usual vector operations and with the norm $\|f\| = \int_T |f| \, dm$ (respectively, $\|f\| = \operatorname{ess\,sup}\{|f(t)| : t \in T\}$).

A set G in a normed linear space E is called a *sun* if, for each $f \in E$ and $g_0 \in P(f)$, we have $g_0 \in P(af + (1 - a) g_0)$ for each $a \geq 1$.

Brosowski [3] proved the following characterization:

2.6. THEOREM. *A set G in a normed linear space is a sun if and only if for each $f \in E \setminus \bar{G}$, $g_0 \in G$ the following statements are equivalent:*

(1) $g_0 \in P(f)$

(2) (g_0, f) *satisfies the Kolmogorov-criterion.*

By Singer [15, Lemma 1.13, p. 83] for $E = L_1(T, m)$ with the property $L_1(T, m)' = L_\infty(T, m)$ a functional L is in $Ep(S_{E'})$ if and only if there exists a function $\beta \in L_\infty(T, m)$ such that $|\beta| = 1$ a.e. on T and $L(f) = \int_T f\beta \, dm$ for each $f \in E$. Thus for $E = L_1(T, m)$ the pair (g_0, f) satisfies the Kolmogorov-criterion (respectively the strict Kolmogorov-criterion) if and only if for each $g \in G$ (respectively $g \in G$, $g \neq g_0$) there exists a $\beta \in L_\infty(T, m)$ such that $|\beta| = 1$ a.e. on T, $\int_T (f - g_0)\beta\, dm = \int_T |f - g_0|\, dm$ and $\int_T (g - g_0)\beta\, dm \leqslant 0$ (respectively $\int_T (g - g_0)\beta\, dm < 0$). Because for a given $g \in G$ we replace T by the union of the supports of f, g_0 and g, which is σ-finite, and therefore we may assume $L_1(T, m)' = L_\infty(T, m)$.

Under application of Theorem 2.6 Deutsch [5] has given the following

2.7. COROLLARY. *Let G be a sun in $L_1(T, m)$, $f \in E \setminus \bar{G}$ and $g_0 \in G$. Then $g_0 \in P(f)$ if and only if for each $g \in G$*

$$\int_{T \setminus Z(f - g_0)} (g - g_0) \operatorname{sgn}(f - g_0)\, dm \leqslant \int_{Z(f - g_0)} |g - g_0|\, dm.$$

Using Theorem 2.6 and Corollary 2.7 we can prove the following characterization of semi-Chebyshev suns in $L_1(T, m)$:

2.8. THEOREM. *Let G be a set in $E = L_1(T, m)$. Then the following statements are equivalent*:

(1) *G is a semi-Chebyshev sun*

(2) *For each $f \in E \setminus \bar{G}$ and $g_0 \in P(f)$ the pair (g_0, f) satisfies the strict Kolmogorov-criterion*

(3) *For each $f \in E \setminus \bar{G}$, $g_0 \in P(f)$, $g \in G$, $g \neq g_0$, there exists a function $\beta \in L_\infty(T, m)$ such that $|\beta| = 1$ a.e. on T,*

$$\int_T (f - g_0)\beta\, dm = \int_T |f - g_0|\, dm \text{ and } \int_T (g - g_0)\, dm < 0$$

(4) For each $f \in E \setminus \bar{G}$, $g_0 \in P(f)$, $g \in G$, $g \neq g_0$,

$$\int_{T \setminus Z(f - g_0)} (g - g_0) \operatorname{sgn}(f - g_0)\, dm < \int_{Z(f - g_0)} |g - g_0|\, dm$$

Proof. The equivalence of (2) and (3) follows from the remark after Theorem 2.6. We show that (4) follows from (1): Assume (4) is not true, then there exist functions $f \in E \setminus \bar{G}$, $g_0 \in P(f)$ and $g_1 \in G$, $g_1 \neq g_0$, such that

$$\int_{T \setminus Z(f - g_0)} (g - g_0) \operatorname{sgn}(f - g_0)\, dm \geqslant \int_{Z(f - g_0)} |g - g_0|\, dm. \qquad \text{(a)}$$

We show that there exists a function $f_0 \in L_1(T, m)$ with $g_0, g \in P(f_0)$ and $g_1 \neq g_0$.

Since $g_0 \in P(f)$ by Corollary 2.7 for each $g \in G$

$$\int_{T \setminus Z(f-g_0)} (g - g_0) \operatorname{sgn}(f - g_0)\, dm \leqslant \int_{Z(f-g_0)} | g - g_0 |\, dm. \tag{b}$$

Combining (a) and (b) it follows

$$\int_{T \setminus Z(f-g_0)} (g_1 - g_0) \operatorname{sgn}(f - g_0)\, dm = \int_{Z(f-g_0)} | g_1 - g_0 |\, dm. \tag{c}$$

We define:

$$f_0(t) := \begin{cases} |(g_1 - g_0)(t)| \operatorname{sgn}(f - g_0)(t) + g_0(t) & \text{if } t \in T \setminus Z(f - g_0), \\ g_0(t) & \text{if } t \in Z(f - g_0), \end{cases}$$

Then it holds:

$$\begin{aligned} \| f_0 - g_0 \| &= \int_T | f - g_0 |\, dm \\ &= \int_{T \setminus Z(f-g_0)} \big| | g_1 - g_0 | \operatorname{sgn}(f - g_0) \big|\, dm \\ &= \int_{T \setminus Z(f-g_0)} | g_1 - g_0 |\, dm. \end{aligned}$$

From (c) it follows:

$$\begin{aligned} \| f_0 - g_1 \| &= \int_T | f_0 - g_1 |\, dm \\ &= \int_{T \setminus Z(f-g_0)} \big| | g_1 - g_0 | \operatorname{sgn}(f - g_0) + g_0 - g_1 \big|\, dm \\ &\quad + \int_{Z(f-g_0)} | g_0 - g_1 |\, dm \\ &= \int_{T \setminus Z(f-g_0)} (| g_1 - g_0 | \operatorname{sgn}(f - g_0) \\ &\quad + (g_0 - g_1)) \operatorname{sgn}(f - g_0)\, dm \\ &\quad + \int_{Z(f-g_0)} | g_0 - g_1 |\, dm \end{aligned}$$

$$= \int_{T\setminus Z(f-g_0)} | g_1 - g_0 | \, dm - \int_{T\setminus Z(f-g_0)} (g_1 - g_0) \operatorname{sgn}(f - g_0) \, dm$$

$$+ \int_{Z(f-g_0)} | g_0 - g_1 | \, dm$$

$$= \int_{T\setminus Z(f-g_0)} | g_1 - g_0 | \, dm = \| f_0 - g_0 \| .$$

Moreover from (b) it follows that for each $g \in G$

$$\| f_0 - g \| = \int_T | f_0 - g | \, dm$$

$$= \int_{T\setminus Z(f-g_0)} || g_1 - g_0 | \operatorname{sgn}(f - g_0) + g_0 - g | \, dm$$

$$+ \int_{Z(f-g_0)} | g_0 - g | \, dm$$

$$\geqslant \int_{T\setminus Z(f-g_0)} (| g_1 - g_0 | \operatorname{sgn}(f - g_0) + g_0 - g) \operatorname{sgn}(f - g_0) \, dm$$

$$+ \int_{Z(f-g_0)} | g_0 - g | \, dm$$

$$= \int_{T\setminus Z(f-g_0)} | g_1 - g_0 | \, dm$$

$$- \int_{T\setminus Z(f-g_0)} (g - g_0) \operatorname{sgn}(f - g_0) \, dm$$

$$+ \int_{Z(f-g_0)} | g_0 - g | \, dm$$

$$\geqslant \int_{T\setminus Z(f-g_0)} | g_1 - g_0 | \, dm = \| f_0 - g_0 \| .$$

Therefore $g_0, g \in P(f)$, $g_1 \neq g_0$. This is a contradiction to G being semi-Chebyshev. Thus (1) implies (4).

We show that (3) follows from (4). If we have (4) then for $g \in G$ we define

$$\beta(t) := \begin{cases} \operatorname{sgn}(f - g_0)(t) & \text{if } t \in T\setminus Z(f - g_0), \\ \operatorname{sgn}(g_0 - g)(t) & \text{if } t \in Z(f - g_0)\setminus Z(g - g_0), \\ 1 & \text{if } t \in Z(f - g_0) \cap Z(g - g_0). \end{cases}$$

Then $|\beta| = 1$ on T and

$$\int_T (f - g_0)\beta\, dm = \int_{T\setminus Z(f-g_0)} (f - g_0)\beta\, dm$$
$$= \int_{T\setminus Z(f-g_0)} (f - g_0)\operatorname{sgn}(f - g_0)\, dm$$
$$= \int_{T\setminus Z(f-g_0)} |f - g_0|\, dm = \int_T |f - g_0|\, dm.$$

$$\int_T (g - g_0)\beta\, dm = \int_{T\setminus Z(f-g_0)} (g - g_0)\operatorname{sgn}(f - g_0)\, dm$$
$$+ \int_{Z(f-g_0)\setminus Z(g-g_0)} (g - g_0)\operatorname{sgn}(g - g_0)\, dm$$
$$+ \int_{Z(f-g_0)\cap Z(g-g_0)} (g - g_0)\, dm$$
$$= \int_{T\setminus Z(f-g_0)} (g - g_0)\operatorname{sgn}(f - g_0)\, dm$$
$$- \int_{Z(f-g_0)} |g - g_0|\, dm$$
$$+ \int_{Z(f-g_0)\cap Z(g-g_0)} (g - g_0)\, dm < 0$$

Thus (4) implies (3).

If we have (2), the fact that G is semi-Chebyshev follows from Lemma 2.1 and that G is a sun follows from Theorem 2.6. Thus (2) implies (1).

Now we will give some examples of semi-Chebyshev sets in $L_1(T, m)$.

First we recall that every convex set in a normed linear space is a sun. An *atom* of a positive measure space (T, m) is a measurable set A in T such that $m(A) > 0$ and for each measurable set B of A either $m(B) = 0$ or $m(A\setminus B) = 0$.

2.9. EXAMPLES. 1. The space $\mathbb{R}^2$ endowed with the norm $\|(x, y)\| = |x| + |y|$ for each $(x, y) \in \mathbb{R}^2$ is a space of type $L_1(T, m)$. It is easy to verify that the set $G = \{(x, y) \in \mathbb{R}^2: x^2 + y^2 \leqslant 1,\ x < 0,\ y < 0\}$ is a non-convex semi-Chebyshev sun in this space.

2. A. L. Garkavi has shown that in $L_1(T, m)$ such that $L_1(T, m)' = L_\infty(T, m)$ there exists a Chebyshev subspace in $L_1(T, m)$ of dimension n (respectively, of codimension n) if and only if (T, m) has at least n atoms (see Singer [10, pp. 233, 331]).

3. Phelps [13] has given an example of a Chebyshev subspace in $L_1(T, m)$

which has neither finite dimension nor finite codimension (see Singer [10, p. 332]). Here it is not necessary that (T, m) contains an atom.

4. Let A be an atom in a positive measure space (T, m) with $m(T\backslash A) > 0$ and $G = \{f \in L_1(T, m): f = 0 \text{ on } T\backslash A, |f(t)| \leqslant 1 \text{ on } A\}$. Then G is a convex Chebyshev set in $L_1(T, m)$: Let f be a function in $L_1(T, m)\backslash G$. If $|f(t)| = 1$ on A, then for $g_f \in G$, defined by $g_f = f$ on A and $g_f = 0$ on $T\backslash A$, it holds:

$$\|f - g_f\| = \int_T |f - g_f| \, dm = \int_{T\backslash A} |f| \, dm < \int_{T\backslash A} |f - g| \, dm$$
$$+ \int_A |g_f - g| \, dm = \int_{T\backslash A} |f - g| \, dm + \int_A |f - g| \, dm$$
$$= \int_T |f - g| \, dm = \|f - g\| \text{ for } g \in G, g \neq g_0 .$$

If $|f(t)| \leqslant 1$ on A, then for $g_f \in G$, defined by $g_f = 1$ on A and $g_f = 0$ on $T\backslash A$, it holds:

$$\|f - g_f\|$$
$$= \int_T |f - g_f| \, dm = \int_A |f - 1| \, dm + \int_{T\backslash A} |f| \, dm$$
$$< \int_A |f - g| \, dm + \int_{T\backslash A} |f| \, dm = \int_A |f - g| \, dm + \int_{T\backslash A} |f - g| \, dm$$
$$= \|f - g\| \text{ for each } g \in G, g \neq g_f .$$

The case $f(t) < -1$ on A can be proved similarily.

5. Let A be an atom as in 4., then $G = \{f \in L_1(T, m): f = 0 \text{ on } T\backslash A, f(t) \geqslant 0 \text{ on } A\}$ is a one-dimensional convex Chebyshev cone in $L_1(T, m)$. This can be shown similarily as in 4.

Theorem 2.4 and Theorem 2.8 snow that it is actually possible to characterize certain semi-Chebyshev sets in $C(X)$ and in a certain sense also in l_A, respectively in $L_1(T, m)$, by the strict Kolmogorov-criterion.

Considering this fact there is the question if it is possible to characterize those elements, which have exactly one best approximation in a non-semi-Chebyshev set, by the strict Kolmogorov-criterion. Examples can easily be constructed to show that this cannot be done, not even for finite-dimensional subspaces in $C(X)$ that are very close to being Chebyshev as e.g. subspaces of spline functions in $C[a, b]$. But in this case we can show that under certain alternation conditions the strict Kolmogorov-criterion is valid.

First some definitions: Let $a = x_0 < x_1 < \cdots < x_k < x_{k+1} = b$ be k

fixed knots in $[a, b]$. The class of the usual *polynomial splines* of degree n with k fixed knots is defined by

$$S_{n,k} := S_{n,k}(x_1, \dots, x_k) = \text{span}\{1, x, \dots, x^n, (x - x_1)_+^n, \dots, (x - x_k)_+^n\}$$

where

$$(x - \xi)_+^n := \begin{cases} 0 & \text{for } x \leqslant \xi \\ (x - \xi)^n & \phantom{\text{for }} x > \xi \end{cases}$$

They form an $n + k + 1$-dimensional subspace of $C[a, b]$. Each function $s \in S_{n,k}$ is in $C^{n-1}[a, b]$ and the restriction of s to the interval $[x_i, x_{i+1}]$, $i = 0, \dots, k$, represents a polynomial of degree n.

It is well known that a function in $C[a, b]$ in general has more than one element of best approximation in $S_{n,k}$.

We need the following restricted interpolation property for spline functions (see Schumaker [14], Karlin [10]).

2.10. THEOREM. *The determinant*

$$\delta\begin{pmatrix} 0, \dots, 0, x_1, \dots, x_k \\ t_1, \dots, t_{n+k+1} \end{pmatrix}$$

$$:= \begin{vmatrix} 1 & t_1 & \cdots & t_1^n & (t_1 - x_1)_+^n & \cdots & (t_1 - x_k)_+^n \\ 1 & t_2 & \cdots & t_2^n & (t_2 - x_1)_+^n & \cdots & (t_2 - x_k)_+^n \\ \vdots & \vdots & & & & & \vdots \\ 1 & t_{n+k+1} & \cdots & t_{n+k+1}^n & (t_{n+k+1} - x_1)_+^n & \cdots & (t_{n+k+1} - x_k)_+^n \end{vmatrix}$$

is nonnegative for all $a \leqslant t_1 < t_2 < \cdots < t_{n+k+1} \leqslant b$ and strictly positive if and only if

$$t_i < x_i < t_{n+i+1}, \qquad i = 1, \dots, k \qquad (a < t_1).$$

Using Theorem 2.10 we can prove the following theorem, which is also true for the more general class of Chebyshevian splinefunctions (for definition see Schumaker [9]).

2.11. THEOREM. *Let $E = C[a, b]$, $G = S_{n,k}$, $f \in E \backslash G$ and $g_0 \in P_G(f)$.* If *there exist $a \leqslant t_1 < \cdots < t_{n+k+2} \leqslant b$ such that*

(1) $t_{i+1} < x_i < t_{n+i+1}$, $i = 1, \dots, k$

(2) $\epsilon(-1)^i (f - g_0)(t_i) = \|f - g_0\|$, $i = 1, \dots, n + k + 2$, $\epsilon = \pm 1$,

then (g_0, f) satisfies the strict Kolmogorov-criterion.

Proof. Assume that the conditions (1) and (2) hold, but that (g_0, f) does not satisfy the strict Kolmogorov-Criterion, i.e., there exists a function $g \in G$, $g \neq 0$, such that, for each $x \in M_{f-g_0}$, we have $(f - g_0)g(x) \geqslant 0$.

Since by (1) there exist points $a \leqslant t_1 < \cdots < t_{n+k+2} \leqslant b$ such that $\epsilon(-1)^i(f - g_0)(t_i) = \| f - g_0 \|$, $i = 1,..., n + k + 2$, $\epsilon = \pm 1$, it follows that $\epsilon(-1)^i g(t_i) \geqslant 0$, $i = 1,..., n + k + 2$, $\epsilon = \pm 1$. From Theorem 2.10 and condition (1) it follows that for each $n + k + 1$ distinct points $u_1,...,u_{n+k+1}$ from $\{t_1,..., t_{n+k+2}\}$ we have

$$\delta\begin{pmatrix}0,..., 0, x_1,..., x_k \\ u_1,..., u_{n+k+1}\end{pmatrix} \neq 0.$$

Therefore there exists a basis $\{g_1,..., g_{n+k+1}\}$ of G such that for each $i \in \{1,..., n + k + 1\}$ we have $g_i(t_j) = 0$, where $1 \leqslant j \leqslant n + k + 1$ and $j \neq i$, and $\epsilon(-1)^i g_i(t_i) = 1$. Then $g = a_1 g_1 + \cdots + a_{n+k+1} g_{n+k+1}$ with $a_1,..., a_{n+k+1} \geqslant 0$ and the scalars a_i are not all zero.

We define

$$D = \begin{vmatrix} g_1(t_1) & \cdots & g_1(t_{n+k+1}) \\ \vdots & & \vdots \\ g_{n+k+1}(t_1) & \cdots & g_{n+k+1}(t_{n+k+1}) \end{vmatrix}$$

and, for each $i \in \{1,..., n + k + 1\}$,

$$D_i = \begin{vmatrix} g_1(t_1) & \cdots & g_1(t_{i-1}) & g_1(t_{i+1}) & \cdots & g_1(t_{n+k+2}) \\ \vdots & & & & & \vdots \\ g_{n+k+1}(t_1) & \cdots & g_{n+k+1}(t_{i-1}) & g_{n+k+1}(t_{i+1}) & \cdots & g_{n+k+1}(t_{n+k+2}) \end{vmatrix}.$$

From Theorem 2.10 it follows that, for each $i \in \{1,..., n + k + 1\}$,

$$DD_i = \epsilon(-1)^{n+k+1} g_i(t_{n+k+2}) \geqslant 0, \quad \text{i.e.,} \quad \epsilon(-1)^{n+k+2} g_i(t_{n+k+2}) \leqslant 0.$$

From this it follows that

$$\begin{aligned} 0 &\leqslant \epsilon(-1)^{n+k+2} g(t_{n+k+2}) \\ &= a_1 \epsilon(-1)^{n+k+2} g_1(t_{n+k+2}) + \cdots + a_{n+k+1} \epsilon(-1)^{n+k+2} g_{n+k+1}(t_{n+k+2}) \leqslant 0. \end{aligned}$$

Then, since $a_1,..., a_{n+k+1} \geqslant 0$, for each $i \in \{1,..., n + k + 1\}$ with

$$a_i \neq 0, \qquad \text{we have} \quad g_i(t_{n+k+2}) = 0.$$

But this shows that, for such an index i, we have $g_i(t_j) = 0$, where $j \in \{1,..., n + k + 2\}$ and $j \neq i$, i.e., $D_i = 0$. Hence

$$\delta\begin{pmatrix} 0,..., 0, x_1,..., x_k \\ t_1,..., t_{i-1}, t_{i+1},..., t_{n+k+2} \end{pmatrix} = 0$$

which, using Theorem 2.10, contradicts condition (1).

3. Strong Unicity of Best Approximations

In this section we apply the unicity results of Section 2 to obtain statements on strong unicity and show that strongly unique elements of best approximation (see Definition 3.1) can be characterized by the strong Kolmogorov-criterion, if the set G is a finite-dimensional subspace in an arbitrary normed linear space.

3.1. Definition. Let G be a set in a normed linear space.

(1) An element $g_0 \in G$ is said to be a *strongly unique element of best approximation* of an element $f \in E$ if there exists a number $K > 0$ such that for each $g \in G$

$$\| f - g \| \geqslant \| f - g_0 \| + K \| g - g_0 \|.$$

(2) G is said to be a *strongly Chebychev* set if each $f \in E$ has a strongly unique element of best approximation in G.

It is easy to see that, in this case, $P_G(f) = \{g_0\}$.

The following lemma proves sufficiency of the strong Kolmogorov-criterion.

3.2. Lemma. *Let E be a normed linear space, G a nonempty set in E, $f \in E \setminus \bar{G}$ and $g_0 \in G$. If (g_0, f) satisfies the strong Kolmogorov-criterion then g_0 is strongly unique element of best approximation of f.*

Proof. According to our assumption, for each $g \in G$, there exists a functional $L_g \in E_{f-g_0}$ with $\operatorname{Re} L_g(g - g_0) \leqslant -K \| g - g_0 \|$. Then for each $g \in G$, $\| f - g \| \geqslant | L_g(f - g) | \geqslant \operatorname{Re} L_g(f - g) + \operatorname{Re} L_g(g - g_0) + K \| g - g_0 \| = \operatorname{Re} L_g(f - g_0) + K \| g - g_0 \| = \| f - g_0 \| + K \| g - g_0 \|$.

Thus g_0 is a strongly unique element of best approximation of f.

3.3. *Remark.* Let E be a normed linear space, G a nonempty set in E, $f \in E \setminus \bar{G}$ and $g_0 \in G$. It is easy to verify that if

$$G(g_0) := \left\{ \frac{g - g_0}{\| g - g_0 \|} : g \in G, g \neq g_0 \right\}$$

is compact and (g_0, f) satisfies the strict Kolmogorov-criterion, then (g_0, f) satisfies the strong Kolmogorov-criterion. Examples can be easily constructed to show that in general $G(g_0)$ is not compact, even if E is finite-dimensional and if G is compact and convex. However, if G is a finite-dimensional subspace or a set with span G is one-dimensional, then $G(g_0)$ is compact for each $g_0 \in G$, and if G is a finite-dimensional convex cone, then $G(0)$ is also compact.

Using Theorem 2.4, Theorem 2.8 and Theorem 2.11 we immediately obtain the following results on strong unicity:

3.4. COROLLARY. *Let G be a nonempty set in $E = I_A$ (respectively in $E = L_1(T, m)$).*

(1) *If G is a finite-dimensional Chebyshev subspace of E then G is a strongly Chebyshev subspace.*

(2) *If G is a one-dimensional convex Chebyshev set in E then G is strongly Chebyshev.*

(3) *If G is a finite-dimensional semi-Chebyshev convex cone of E then for each $f \in E$ with $0 \in P(f)$ the element 0 is a strongly unique element of best approximation of f.*

Statement (1) in Corollary 3.4 has been proved by Newman, Shapiro [12] for $E = I_\phi = C(X)$, by Ault, Deutsch, Morris, Olson [1] for $E = I_{\{\infty\}} = C_0(T)$ and by Wulbert [17] for $E = L_1(T, m)$.

3.5. COROLLARY. *In Theorem 2.11 the element g_0 is a strongly unique element of best approximation of f.*

Schumaker [14] has shown that in Theorem 2.11 the element g_0 is the unique element of best approximation of f.

Now we will show that strongly unique elements of best approximation can be characterized by the strong Kolmogorov-criterion. For this we need the following characterization of strongly unique elements of best approximation due to Wulbert [17] for real normed linear spaces and due to Bartelt, McLaughlin [2] for complex normed linear spaces:

3.6. THEOREM. *Let G be a linear subspace of a normed linear space E. An element $g_0 \in G$ is a strongly unique element of best approximation of an element $f \in E \setminus \bar{G}$ if and only if there exists a number $K > 0$ such that, for each $g \in G$,*

$$\min\{\operatorname{Re} L(g) : L \in S_{f-g_0}\} \leqslant -K \| g \| .$$

Using standard arguments (see Köthe [11] and Brosowski [3], Lemma 2) from Theorem 3.6 we obtain the following

3.7. COROLLARY. *Let G be a linear subspace of a normed linear space E. An element $g_0 \in G$ is a strongly unique element of best approximation of an element $f \in E \backslash \bar{G}$ if and only if (g_0, f) satisfies the strong Kolmogorov-criterion.*

Corollary 3.7 has been proved by Bartelt, McLaughlin [2] for finite-dimensional subspaces of $C(X)$, where the functions in $C(X)$ are complex-valued.

3.8. *Remark.* Now we can see that it is not possible to characterize finite-dimensional Chebyshev subspaces in an arbitrary normed linear space by the strict Kolmogorov-criterion. Because would this be true then from Remark 3.3 and Theorem 3.8 it would follow that each finite-dimensional Chebyshev subspace in an arbitrary normed linear space is strongly Chebyshev. Wulbert [17], however, has shown that in a smooth normed linear space no Chebyshev subspace is strongly Chebyshev.

3.9. DEFINITION. For a nonempty set G in a normed linear space E the metric projection $P: E \to 2^G$ is called *pointwise Lipschitzian at $f_0 \in E$*, if $P(f_0) = \{g_{f_0}\}$ and if there exists a number $\bar{K} > 0$ such that for each $f \in E$ and each $g_f \in P(f)$

$$\| g_{f_0} - g_f \| \leqslant \bar{K} \| f_0 - f \| .$$

We say $P: E \to G$ is *pointwise Lipschitzian* if P is pointwise Lipschitzian at each $f_0 \in F$.

The following lemma, which is due to Cheney [4, p. 82], shows that pointwise Lipschitzian continuity of the metric projection follows from strong unicity properties:

3.10. LEMMA. *Let G be a set in a normed linear space E. If $g_0 \in E$ is a strongly unique element of best approximation of an element $f_0 \in E$ then the metric projection $P: E \to 2^G$ is pointwise Lipschitzian at $f_0 \in E$.*

Using Lemma 3.10 we immediately get from Corollary 3.4 and Corollary 3.5 the following statements on pointwise Lipschitzian metric projections:

3.11 COROLLARY. *Let G be a nonempty set in $E = l_A$ (respectively in $E = L_1(T, m)$).*

(1) *If G is a finite-dimensional Chebyshev subspace of E then the metric projection $P: E \to G$ is pointwise Lipschitzian.*

(2) *If G is a one-dimensional convex Chebyshev set in E. Then the metric projection $P: E \to G$ is pointwise Lipschitzian.*

(3) *If G is a finite-dimensional convex semi-Chebyshev cone of E then for each $f_0 \in E$ with $0 \in P(f_0)$ the metric projection $P: E \to G$ is pointwise Lipschitzian at f_0.*

A direct proof of statement (1) in Corollary 3.11 has been given by Freud [8] for $E = C[a, b]$.

3.12. COROLLARY. *In Theorem* 2.11 *the metric projection* $P: E \to 2^G$ *is pointwise Lipschitzian at* f.

Corollary 3.12 has been proved by Schumaker [14].

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